MTH 203: Introduction to Groups and Symmetry

Homework VIII

(Due 17/11/2022)

Problems for submission

1. Consider the set of 8 symbols
$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$
with a product operation satisfying the following sets of relations:
   - $i^2 = j^2 = k^2 = -1$.
   - $ij = k$, $jk = i$, $ki = j$. (or equivalently, $ijk = -1$.)
   - $(-1)^2 = 1$.

   (a) Show that relations above induce a binary operation on Q_8 under which it forms a non-abelian group called the *group of quaternions.*

   (b) Show that Q_8 has a unique subgroup of order 2 given by $\{\pm 1\}$, which is also its center.

   (c) Show that Q_8 has exactly three distinct subgroups of order 4, all of which are cyclic, namely:
$$\langle i \rangle = \{\pm 1, \pm i\},\ \langle j \rangle = \{\pm 1, \pm j\},\ \text{and } \langle k \rangle = \{\pm 1, \pm k\}.$$

   (d) Show that Q_8 is not isomorphic to D_8.

Problems for practice

1. Let $R_{x,\theta} \in \mathrm{Sym}(\mathbb{R}^2)$ denote the counterclockwise rotation about a point $x \in \mathbb{R}^2$ by an angle θ. Show that for rotations $R_{x_1,\theta_1}, R_{x_2,\theta_2} \in \mathrm{Sym}(\mathbb{R}^2)$ such that $\theta_1 + \theta_2 = 2k\pi$ for some $k \in \mathbb{Z}$, the product $R_{x_1,\theta_1} R_{x_2,\theta_2} \in \mathrm{Sym}(\mathbb{R}^2)$ is a translation.

2. Let $\mathbb{R}_n[x]$ be the additive group of all polynomials of degree $\leq n$ in the variable x with coefficients from $\mathbb{R}$. For $1 \leq k \leq n$, let $D_k : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ be the k^{th} derivative map defined by
$$D_k(p(x)) = \frac{d^k}{dx^k}(p(x)),\ \forall\, p(x) \in \mathbb{R}_n[x].$$

   (a) Show that D_k is a homomorphism.

   (b) Determine $\mathrm{Ker}\, D_k$ and $\mathrm{Im}\, D_k$.

   (c) Show that $\mathbb{R}_n[x]/\mathbb{R}_{n-1}[x] \cong \mathbb{R}$.

3. Consider the group $G = A_4$.

   (a) Describe the order 2 subgroups of G.

(b) Describe the order 3 subgroups of G.

(c) Does G have an element g with $o(g) \geq 4$? Explain why, or why not.

(d) Show that G has a unique subgroup of order 4.

4. (a) Is the group $\mathrm{SO}(2, \mathbb{R})$ abelian? Prove or disprove.

(b) Describe two distinct monomorphisms $\mathrm{SO}(2, \mathbb{R}) \to \mathrm{SO}(3, \mathbb{R})$.

(c) Show that $\mathrm{SO}(3, \mathbb{R})$ is non-abelian.

5. Let G be a finite group of order n.

(a) Show that for each $g \in Z(G)$, the conjugacy class $[g]_c = \{g\}$.

(b) Let $g_1, \ldots, g_k$ be the the representatives of the distinct conjugacy classes in $G \setminus Z(G)$. Show that

$$n = |Z(G)| + \sum_{i=1}^{k} |[g_i]_c|.$$

(c) Suppose that $n = p^2$, where p is prime. Assuming the fact that $p \mid |[g_i]_c|$, for each i, show that G is abelian.